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Research on Fractional Differential Problem of Two Types of Fractional Analytic Functions

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Abstract: **This paper provides the formulas of arbitrary order fractional derivative of two types of fractional analytic functions. Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions play important roles in this article. In fact, our results are generalizations of the results of ordinary calculus.**

Keywords: **Fractional derivative, fractional analytic functions, Jumarie type of R-L fractional derivative, new multiplication.**

I. INTRODUCTION

Fractional calculus originated in 1695 and almost at the same time as traditional calculus. Fractional calculus is considered to be a useful tool for understanding and simulating many natural and artificial phenomena. It has developed rapidly in different scientific fields in the past few decades, including not only mathematics and physics, but also engineering, biology, economics and chemistry [1-13].

However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [14-18]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, we obtain the formulas of any order fractional derivative of two types of fractional analytic functions:

$$\left({}_0D_x^{\alpha}\right)^n\left[E_\alpha(px^\alpha)\otimes_\alpha cos_\alpha(qx^\alpha)\right]. \tag{1}$$

And

$$\left({}_0D_x^{\alpha}\right)^n\left[E_\alpha(px^\alpha)\otimes_\alpha sin_\alpha(qx^\alpha)\right]. \tag{2}$$

Where $0<\alpha\leq 1$, n is any positive integer, p,q are real numbers, $p^2+q^2\neq 0$. Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our results are generalizations of traditional calculus results.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper and its properties.

Definition 2.1 ([19])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\,. \tag{3}$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer n, we define $\left({}_{x_0}D_x^\alpha\right)^n[f(x)] = \left({}_{x_0}D_x^\alpha\right)\left({}_{x_0}D_x^\alpha\right)\cdots\left({}_{x_0}D_x^\alpha\right)[f(x)]$, the n-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([20])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^\alpha\right)\left[(x-x_0)^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_{x_0}D_x^\alpha\right)[C] = 0. \tag{5}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3 ([21])**:** Suppose that x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([22])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{7}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{8}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k}. \end{aligned} \tag{9}$$

Definition 2.5 ([23])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k}, \tag{10}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha k}. \tag{11}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha k}, \tag{12}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha k}. \tag{13}$$

Definition 2.6 ([24])**:** If $0 < \alpha \le 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}. \tag{14}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2k}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2k+1)}. \tag{16}$$

Definition 2.7 ([25])**:** Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha (-1)}$.

Definition 2.8: If the complex number $z = p + iq$, where p, q are real numbers, and $i = \sqrt{-1}$. p, the real part of z, is denoted by $\mathrm{Re}(z)$; q the imaginary part of z, is denoted by $\mathrm{Im}(z)$.

III. MAIN RESULTS AND EXAMPLES

In this section, we obtain any order fractional derivative of two types of fractional analytic functions. On the other hand, some examples are given to illustrate our results. At first, we need a lemma.

Lemma 3.1: *If n is any positive integer and p, q are real numbers, $p^2 + q^2 \ne 0$. Then*

$$(p + iq)^n = \left(\sqrt{p^2 + q^2}\right)^n \cdot [cos(n\theta) + isin(n\theta)]. \tag{17}$$

$$Where\ \theta = \begin{cases} arctan\left(\frac{q}{p}\right) & if\ p \ne 0, \\ \frac{\pi}{2} & if\ p = 0, q > 0, \\ -\frac{\pi}{2} & if\ p = 0, q < 0. \end{cases}$$

Proof

$$\begin{aligned} &(p + iq)^n \\ &= \left(\sqrt{p^2 + q^2} \cdot \left(\frac{p}{\sqrt{p^2+q^2}} + i\frac{q}{\sqrt{p^2+q^2}}\right)\right)^n \\ &= \left(\sqrt{p^2 + q^2}\right)^n \cdot \left(\frac{p}{\sqrt{p^2+q^2}} + i\frac{q}{\sqrt{p^2+q^2}}\right)^n \\ &= \left(\sqrt{p^2 + q^2}\right)^n \cdot (cos\theta + isin\theta)^n \\ &= \left(\sqrt{p^2 + q^2}\right)^n \cdot [cos(n\theta) + isin(n\theta)]. \end{aligned}$$

Q.e.d.

Theorem 3.2: *Let $0 < \alpha \le 1$, n be any positive integer, p, q be real numbers, $p^2 + q^2 \ne 0$. Then*

$$\left({}_0D_x^\alpha\right)^n\left[E_\alpha(px^\alpha) \otimes_\alpha cos_\alpha(qx^\alpha)\right] = \left(\sqrt{p^2 + q^2}\right)^n \cdot \left[E_\alpha(px^\alpha) \otimes_\alpha [cos(n\theta) \cdot cos_\alpha(qx^\alpha) - sin(n\theta) \cdot sin_\alpha(qx^\alpha)]\right]. \tag{18}$$

$$Where\ \ \theta = \begin{cases} arctan\left(\frac{q}{p}\right) & if\ p \ne 0, \\ \frac{\pi}{2} & if\ p = 0, q > 0, \\ -\frac{\pi}{2} & if\ p = 0, q < 0. \end{cases}$$

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Proof By Lemma 3.1,

$$\begin{aligned}
&\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[E_{\alpha}(px^{\alpha})\otimes_{\alpha} cos_{\alpha}(qx^{\alpha})\right]\\
&=\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[\mathrm{Re}(E_{\alpha}((p+iq)x^{\alpha}))\right]\\
&=\mathrm{Re}\left(\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[E_{\alpha}((p+iq)x^{\alpha})\right]\right)\\
&=\mathrm{Re}\left((p+iq)^{n}\cdot E_{\alpha}((p+iq)x^{\alpha})\right)\\
&=\mathrm{Re}\left(\left(\sqrt{p^2+q^2}\right)^{n}\cdot[cos(n\theta)+isin(n\theta)]\cdot E_{\alpha}(px^{\alpha})\otimes_{\alpha}[cos_{\alpha}(qx^{\alpha})+isin_{\alpha}(qx^{\alpha})]\right)\\
&=\left(\sqrt{p^2+q^2}\right)^{n}\cdot\left[E_{\alpha}(px^{\alpha})\otimes_{\alpha}[cos(n\theta)\cdot cos_{\alpha}(qx^{\alpha})-sin(n\theta)\cdot sin_{\alpha}(qx^{\alpha})]\right].
\end{aligned}$$

Q.e.d.

Theorem 3.3: *If* $0<\alpha\leq 1$, n *is any positive integer*, p, q *are real numbers*, $p^2+q^2\neq 0$. *Then*

$$\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[E_{\alpha}(px^{\alpha})\otimes_{\alpha} sin_{\alpha}(qx^{\alpha})\right]=\left(\sqrt{p^2+q^2}\right)^{n}\cdot\left[E_{\alpha}(px^{\alpha})\otimes_{\alpha}[cos(n\theta)\cdot sin_{\alpha}(qx^{\alpha})+sin(n\theta)\cdot cos_{\alpha}(qx^{\alpha})]\right]. \tag{19}$$

$$\text{Where } \theta=\begin{cases} arctan\left(\frac{q}{p}\right) & if\ p\neq 0,\\ \frac{\pi}{2} & if\ p=0, q>0,\\ -\frac{\pi}{2} & if\ p=0, q<0.\end{cases}$$

Proof Using Lemma 3.1 yields

$$\begin{aligned}
&\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[E_{\alpha}(px^{\alpha})\otimes_{\alpha} sin_{\alpha}(qx^{\alpha})\right]\\
&=\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[\mathrm{Im}(E_{\alpha}((p+iq)x^{\alpha}))\right]\\
&=\mathrm{Im}\left(\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[E_{\alpha}((p+iq)x^{\alpha})\right]\right)\\
&=\mathrm{Im}\left((p+iq)^{n}\cdot E_{\alpha}((p+iq)x^{\alpha})\right)\\
&=\mathrm{Im}\left(\left(\sqrt{p^2+q^2}\right)^{n}\cdot[cos(n\theta)+isin(n\theta)]\cdot E_{\alpha}(px^{\alpha})\otimes_{\alpha}[cos_{\alpha}(qx^{\alpha})+isin_{\alpha}(qx^{\alpha})]\right)\\
&=\left(\sqrt{p^2+q^2}\right)^{n}\cdot\left[E_{\alpha}(px^{\alpha})\otimes_{\alpha}[cos(n\theta)\cdot sin_{\alpha}(qx^{\alpha})+sin(n\theta)\cdot cos_{\alpha}(qx^{\alpha})]\right].
\end{aligned}$$

Q.e.d.

Example 3.4: Suppose that $0<\alpha\leq 1$, then

$$\begin{aligned}
&\left({}_{0}D_{x}^{\alpha}\right)^{17}\left[E_{\alpha}(2x^{\alpha})\otimes_{\alpha} cos_{\alpha}(3x^{\alpha})\right]\\
&=\left(\sqrt{13}\right)^{17}\cdot\left[E_{\alpha}(2x^{\alpha})\otimes_{\alpha}\left[cos\left(17\cdot arctan\left(\frac{3}{2}\right)\right)\cdot cos_{\alpha}(3x^{\alpha})-sin\left(17\cdot arctan\left(\frac{3}{2}\right)\right)\cdot sin_{\alpha}(3x^{\alpha})\right]\right].
\end{aligned}\tag{20}$$

And

$$\begin{aligned}
&\left({}_{0}D_{x}^{\alpha}\right)^{11}\left[E_{\alpha}(4x^{\alpha})\otimes_{\alpha} sin_{\alpha}(5x^{\alpha})\right]\\
&=\left(\sqrt{41}\right)^{11}\cdot\left[E_{\alpha}(4x^{\alpha})\otimes_{\alpha}\left[cos\left(11\cdot arctan\left(\frac{5}{4}\right)\right)\cdot sin_{\alpha}(5x^{\alpha})+sin\left(11\cdot arctan\left(\frac{5}{4}\right)\right)\cdot cos_{\alpha}(5x^{\alpha})\right]\right].
\end{aligned}\tag{21}$$

IV. CONCLUSION

In this paper, the formulas of any order fractional derivative of two types of fractional analytic functions are obtained. Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions play important roles in this article. In fact, our results are generalizations of classical calculus results. In the future, we will continue to study the problems in engineering mathematics and fractional differential equations by using our methods.

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